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Absence of Species Doubling in Finite-Element Quantum Electrodynamics

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Abstract

In this letter it will be demonstrated explicitly that the finite-element formulation of quantum electrodynamics is free from fermion doubling. We do this by (1) examining the lattice fermion propagator and using it to compute the one-loop vacuum polarization on the lattice, and (2) by an explict computation of vector and axial-vector current anomalies for an arbitrary rectangular lattice in the Schwinger model. There it is shown that requiring that the vector current be conserved necessitates the use of a square lattice, in which case the axial-vector current is anomalous.

I. INTRODUCTION

The application of the finite-element method to the Heisenberg equations of motion of a quantum theory was proposed over a decade ago [1]. It was immediately recognized that the formulation was unitary, in that canonical commutation relations are maintained at each lattice site. (This was in addition to the fact that of all possible discretizations of the equations of motion, the finite-element prescription is the most accurate [2].) The Dirac

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equation was studied shortly thereafter [3], and it was found that the dispersion relation did not admit fermion doubling, usually present in lattice formulations. The reason that the present formulation could evade the no-go theorems [4] is that no local Hamiltonian in the Schrödinger sense exists. However, this conclusion has remained somewhat controversial, and it is the purpose of the present letter to offer new evidence for the absence of species doublers. In particular in Sec. II we display the free fermion propagator, and show that only a single fermion is represented, and that when these propagators are used in a dynamical loop calculation, the anomaly in the Schwinger model is indeed recovered. In Sec. III we explicitly compute the divergence of the vector current in $(1+1)$ dimensions, and show for a square lattice that the vector current is conserved, while the axial-vector current is anomalous. We repeat the calculation for small rectangular lattices, and show that only for square lattice case can a vector anomaly be avoided.

Abelian and non-Abelian gauge theories in the finite-element context were discussed, respectively, in [5] and [6]. The Schwinger model was treated in this context previously in [5,7]. A review of the entire program appears in [8].

II. FREE LATTICE FERMION PROPAGATOR. ONE-LOOP CALCULATION OF VACUUM POLARIZATION

From the free finite-element lattice Dirac equation,

$$\frac{i\gamma^0}{h}(\psi_{\overline{\mathbf{m}},n+1}-\psi_{\overline{\mathbf{m}},n})+\frac{i\gamma^j}{\Delta}(\psi_{m_j+1,\overline{\mathbf{m}}_\perp,\overline{n}}-\psi_{m_j,\overline{\mathbf{m}}_\perp,\overline{n}})-\mu\psi_{\overline{\mathbf{m}},\overline{n}}=0, \tag{2.1}$$

where μ is the electron mass, h is the temporal lattice spacing, Δ is the spatial lattice spacing, $\mathbf{m}$ represents a spatial lattice coordinate, n a temporal coordinate, and overbars signify forward averaging:

$$x_{\overline{m}}=\frac{1}{2}(x_{m+1}+x_m), \tag{2.2}$$

it is easy to derive the free fermion propagator [9],

$$G_{\mathbf{m},n;\mathbf{m}',n'} = \frac{h}{4\pi}\int_{-\pi/h}^{\pi/h} d\hat{\Omega} e^{-ih\hat{\Omega}(n-n')}\frac{1}{L^3}\sum_{\mathbf{p}} e^{i\mathbf{p}\cdot(\mathbf{m}-\mathbf{m}')2\pi/M}$$
$$\times \frac{\gamma^0 \sin h\hat{\Omega} + (\mu - \boldsymbol{\gamma}\cdot\tilde{\mathbf{p}})h\cos^2 h\hat{\Omega}/2}{\cos h(\Omega - i\epsilon) - \cos h\hat{\Omega}}. \tag{2.3}$$

Here $L = M\Delta$ is the length of the spatial lattice, and

$$\tilde{\mathbf{p}} = \frac{2\mathbf{t}}{\Delta}, \quad \omega = \tilde{p}^0 = \sqrt{\tilde{\mathbf{p}}^2 + \mu^2}, \quad (\mathbf{t}_{\mathbf{p}})_i = \tan p_i\pi/M. \tag{2.4}$$

The mass-shell "energy" Ω is defined in terms of λ, the eigenvalue of the Dirac transfer matrix,

$$\lambda = \frac{1 + ih\omega/2}{1 - ih\omega/2} \equiv e^{i\Omega(h)h}. \tag{2.5}$$

Notice that we may solve (2.5) for ω:

$$\omega = \frac{2}{h}\tan\frac{h\Omega}{2}. \tag{2.6}$$

We have taken M, the number of lattice points in a given spatial direction, to be odd, so that ψ is periodic on the spatial lattice.

We note that the singularities in the fermion propagator (2.3) are unique: That is, for a given lattice momentum $\tilde{\mathbf{p}}$ there is a pole at a single energy $\hat{\Omega}$ between 0 and π/h. A species doubler would be signalled by having the minimum value of ω, μ, occur at $\mathbf{p} = 0(\text{mod } M/2)$ not $0(\text{mod } M)$.

We confirm this conclusion by presenting the results of a dynamical look calculation based on the propagator (2.3) of the photon polarization tensor in the Schwinger model ($\mu = 0$, $d = 2$) [For details of this calculation, see [9].] In the continuum, the polarization tensor is transverse,

$$\Pi^{\mu\nu} = \Pi(k^2)\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right), \tag{2.7}$$

where an easy calculation shows that $\Pi(0) = -e^2/\pi$, which gives the boson mass in the model. It is straightforward to carry out the corresponding calculation using the lattice propagator; the result is shown in Figure 1. The Schwinger model corresponds to $\mu = 0$.

In fact, it is clear that for very small $\nu = \mu\Delta$ the lattice calculation diverges, which is as expected when $\nu \sim M^{-1}$. There is, for all values of $r = h/\Delta$, a strong shoulder at the continuum value of $\Pi(0)$. (The sensitivity as $r \to 1$ is a lattice artifact.) Such behavior is completely consistent with the absence of species doubling.

III. INTERACTIONS. AXIAL-VECTOR ANOMALY

Interactions of an electron with a background electromagnetic field is given in terms of a transfer matrix T:

$$\psi_{n+1} = T_n \psi_n, \tag{3.1}$$

which is to be understood as a matrix equation in $\overline{\mathbf{m}}$. Explicitly, in the gauge $A^0 = 0$, [10]

$$T = 2U^{-1} - 1, \quad U = 1 + \frac{ih\mu\gamma^0}{2} - \frac{h}{\Delta}\gamma^0 \boldsymbol{\gamma} \cdot \boldsymbol{\mathcal{D}}, \tag{3.2}$$

where [9]

$$\mathcal{D}^j_{\mathbf{m},\mathbf{m}'} = -(-1)^{m_j + m'_j} [\epsilon_{m_j,m'_j} \cos\hat{\zeta}_{m_j,m'_j} - i \sin\hat{\zeta}_{m_j,m'_j}] \sec\zeta_{(j)}\, \delta_{\mathbf{m}_\perp,\mathbf{m}'_\perp}. \tag{3.3}$$

Here

$$\epsilon_{m,m'} = \begin{cases} 1, & m > m', \\ 0, & m = m', \\ -1, & m < m', \end{cases} \tag{3.4}$$

and (the following are local and unaveraged in $\mathbf{m}_\perp$, n)

$$\zeta_{m_j} = \frac{e\Delta}{2} A^j_{\overline{m_j - 1}}, \quad \zeta_{(j)} = \sum_{m_j=1}^{M} \zeta_{m_j}, \tag{3.5a}$$

and

$$\hat{\zeta}_{m_j,m'_j} = \sum_{m''_j=1}^{M} \mathrm{sgn}\,(m''_j - m_j)\mathrm{sgn}\,(m''_j - m'_j)\zeta_{m''_j}, \tag{3.5b}$$

with

$$\mathrm{sgn}\,(m-m') = \epsilon_{m,m'} - \delta_{m,m'}. \tag{3.6}$$

Because $\mathcal{D}$ is anti-Hermitian, it follows that T is unitary, that is, that $\phi_{\mathbf{m},n} = \psi_{\overline{\mathbf{m}},n}$ is the canonical field variable satisfying the canonical anticommutation relations

$$\{\phi_{\mathbf{m},n}, \phi^\dagger_{\mathbf{m}',n}\} = \frac{1}{\Delta^3}\delta_{\mathbf{m},\mathbf{m}'}. \tag{3.7}$$

In this letter we will consider the Schwinger model, that is, the case with dimension $d=2$ and mass $\mu=0$. Because the light-cone aligns with the lattice in that case, we first set $h=\Delta$. Then we see that the transfer matrix for positive or negative chirality, that is, for the eigenvalue of $i\gamma_5 = \gamma^0\gamma^1$ equal to ± 1, is

$$T_\pm = \frac{1\pm\mathcal{D}}{1\mp\mathcal{D}}. \tag{3.8}$$

From (3.3) we see that the numerator of T is

$$(1\pm\mathcal{D})_{m,m'} = [\delta_{m,m'}e^{\pm i\zeta} \mp (-1)^{m+m'}\epsilon_{m,m'}e^{-i\epsilon_{m,m'}\hat{\zeta}_{m,m'}}]\sec\zeta, \tag{3.9}$$

while it is a simple calculation to verify that the inverse of this operator is

$$(1+\mathcal{D})^{-1}_{m,m'} = \frac{1}{2}\left(\delta_{m,m'} + \delta_{m,m'-1}e^{-2i\zeta_{m'}}\right), \tag{3.10a}$$

$$(1-\mathcal{D})^{-1}_{m,m'} = \frac{1}{2}\left(\delta_{m,m'} + \delta_{m,m'+1}e^{2i\zeta_{m}}\right). \tag{3.10b}$$

It is therefore immediate to find

$$(T_+)_{m,m'} = \delta_{m,m'+1}e^{2i\zeta_m}, \tag{3.11a}$$

$$(T_-)_{m,m'} = \delta_{m+1,m'}e^{-2i\zeta_{m'}}, \tag{3.11b}$$

which simply says that the $+$ $(-)$ chirality fermions move on the light-cone to the right (left), acquiring a phase proportional to the vector potential. Solution (3.11) directly implies the chiral anomaly in the Schwinger model, as shown in [5]. We will now expand on and generalize that calculation.

First, the canonical anticommutation relations (3.7) show that we can write the momentum expansion of the (free) Dirac field as (if M is even we replace p by $p+1/2$ in the exponent)

$$\phi_{\mathbf{m},n} = \sum_{s,\mathbf{p}} \sqrt{\frac{\mu}{\omega}} (b_{\mathbf{p},s} u_{\mathbf{p},s} \lambda^{-n} e^{i\mathbf{p}\cdot\mathbf{m}2\pi/M} + d^\dagger_{\mathbf{p},s} v_{\mathbf{p},s} \lambda^n e^{-i\mathbf{p}\cdot\mathbf{m}2\pi/M}), \tag{3.12}$$

where the spinors are normalized according to

$$\sum_s \tilde{u}_\pm \tilde{u}_\pm^\dagger \gamma^0 = \mp \frac{\mu \pm \gamma \cdot \tilde{p}}{2\mu} \equiv \pm\Lambda_\pm, \tag{3.13}$$

where $u = \tilde{u}_-$ and $v = \tilde{u}_+$. The creation and annihilation operators satisfy ($d = 4$)

$$\{b_{\mathbf{p},s}, b^\dagger_{\mathbf{p}',s'}\} = \frac{1}{L^3} \delta_{\mathbf{p},\mathbf{p}'} \delta_{s,s'}, \tag{3.14a}$$

$$\{d_{\mathbf{p},s}, d^\dagger_{\mathbf{p}',s'}\} = \frac{1}{L^3} \delta_{\mathbf{p},\mathbf{p}'} \delta_{s,s'}, \tag{3.14b}$$

and all other anticommutators of these operators vanish. It is then easy to verify that for M even, for the particular case of $d = 2$ and $\mu = 0$, that in the Fock-space vacuum[1]

$$\langle \phi^{(\pm)\,\dagger}_{m,n} \phi^{(\pm)}_{m',n} \rangle = \mp \frac{i}{2L} \frac{1 - (-1)^{m-m'}}{\sin(m - m')\pi/M}, \tag{3.15a}$$

while for M odd,

$$\langle \phi^{(\pm)\,\dagger}_{m,n} \phi^{(\pm)}_{m',n} \rangle = \mp \frac{i}{2L} \frac{\cos(m - m')\pi/M - (-1)^{m-m'}}{\sin(m - m')\pi/M}. \tag{3.15b}$$

In both cases, the vacuum expectation value is taken to zero if $m = m'$.

It is then quite straightforward to compute the vacuum expectation value of the finite-element divergence of the the vector current. On the lattice the gauge-invariant current is written in terms of the all-averaged field Ψ,

$$\Psi_{\mathbf{m},n} = \psi_{\overline{\mathbf{m}},\overline{n}} = \frac{1}{2}(\phi_{\mathbf{m},n+1} + \phi_{\mathbf{m},n}), \tag{3.16}$$

rather than the canonical field ϕ. That is, the current is

$$J^\mu_{\mathbf{m},n} = e \Psi^\dagger_{\mathbf{m},n} \gamma^0 \gamma^\mu \Psi_{\mathbf{m},n} = \frac{e}{4} [\phi^\dagger_n (1 + T^\dagger_n)]_{\mathbf{m}} \gamma^0 \gamma^\mu [(1 + T_n)\phi_n]_{\mathbf{m}}. \tag{3.17}$$

[1]In the following we omit terms arising from reordering the operators on the left-hand side of (3.15) using the anticommutator (3.7); these are local in m, m', and hence do not affect the $r = 1$ results. In general they only contribute an extraneous spatial constant to $\langle J^0 \rangle$.

(For a discussion of why this choice of current is used, see the Appendix of [9].) In the absence of interactions it is easy to show that

$$\langle \text{“}\partial_\mu J^\mu\text{”} \rangle = 0, \tag{3.18}$$

where the quotation marks signify a finite-element lattice divergence and the brackets represent a Fock-space vacuum expectation value. In two dimensions, that divergence is

$$\begin{aligned} \text{“}\partial^\mu J_\mu\text{”} &= \frac{e}{2h}(\psi^\dagger_{\overline{m}+1,\overline{n}+1}\psi_{\overline{m}+1,\overline{n}+1} + \psi^\dagger_{\overline{m},\overline{n}+1}\psi_{\overline{m},\overline{n}+1} - \psi^\dagger_{\overline{m}+1,\overline{n}}\psi_{\overline{m}+1,\overline{n}} - \psi^\dagger_{\overline{m},\overline{n}}\psi_{\overline{m},\overline{n}}) \\ &+ \frac{e}{2\Delta}(\psi^\dagger_{\overline{m}+1,\overline{n}+1}i\gamma_5\psi_{\overline{m}+1,\overline{n}+1} + \psi^\dagger_{\overline{m}+1,\overline{n}}i\gamma_5\psi_{\overline{m}+1,\overline{n}} - \psi^\dagger_{\overline{m},\overline{n}+1}i\gamma_5\psi_{\overline{m},\overline{n}+1} - \psi^\dagger_{\overline{m},\overline{n}}i\gamma_5\psi_{\overline{m},\overline{n}}). \end{aligned} \tag{3.19}$$

For the case of a square lattice, $h = \Delta$, this is simply expressed in terms of eigenvectors of $i\gamma_5$, which in turn may be expressed in terms of the canonical field $\phi_{m,n+1}$ at the intermediate time. A short calculation reveals that

$$\text{“}\partial^\mu J_\mu\text{”} = A^{(+)} + A^{(-)}, \tag{3.20}$$

where

$$\begin{aligned} A^{(\pm)} = \frac{e}{4h}\Big[&\phi^{(\pm)\dagger}_{m,n+1}\phi^{(\pm)}_{m+1,n+1}e^{-i(\zeta_{m+1,n}+\zeta_{m+1,n+1})} \\ &- \phi^{(\pm)\dagger}_{m+1,n+1}\phi^{(\pm)}_{m,n+1}e^{i(\zeta_{m+1,n}+\zeta_{m+1,n+1})}\Big]\, 2i\sin(\zeta_{m+1,n} - \zeta_{m+1,n+1}) \end{aligned} \tag{3.21}$$

The vacuum expectation value of $A^{(+)}$ cancels that of $A^{(-)}$, and hence the vector current is conserved, because (q is the eigenvalue of $i\gamma_5$)

$$\langle \phi^{(q)\dagger}_{m\pm1}\phi^{(q)}_m \rangle = \mp q\frac{i}{L}\left\{\begin{array}{c} \cos^2\pi/2M \\ 1 \end{array}\right\}\csc\pi/M, \tag{3.22}$$

for M odd or even respectively. On the other hand, the divergence of the axial-vector current,

$$\text{“}\partial_\mu J^{\mu}_5\text{”} = A^{(+)} - A^{(-)}, \tag{3.23}$$

is not zero:

$$\langle ``\partial_\mu J_5^\mu" \rangle = \frac{2e}{hL}\frac{1}{\sin\pi/M}\left\{\begin{array}{c}\cos^2\pi/2M\\ \\ 1\end{array}\right\}\sin(\zeta_{m+1,n+1}-\zeta_{m+1,n})\cos(\zeta_{m+1,n+1}+\zeta_{m+1,n}). \tag{3.24}$$

In fact, expanding this in powers of ehA, we find (because $E = \dot{A}$)

$$\langle ``\partial_\mu J_5^\mu" \rangle = \frac{e^2}{M\sin\pi/M}\left\{\begin{array}{c}\cos^2\pi/2M\\ \\ 1\end{array}\right\}E\left(1++O((ehA)^2)\right). \tag{3.25}$$

For $M = 2$ the error in the coefficient relative to the continuum value of the coefficient, e^2/π is about 50%, while for $M = 3$ the error drops to about 10%; in general, the relative error is of order M^{-2}.

We now consider a rectangular lattice, $h \neq \Delta$. Because this case is rather more complicated that the square lattice considered above, we content ourselves with the two smallest possible lattices, $M = 2$ and $M = 3$. For the $M = 2$ case the transfer matrix

$$T_\pm = \frac{1 \pm r\mathcal{D}}{1 \mp r\mathcal{D}}, \tag{3.26}$$

where $r = h/\Delta$, is easily found to be (the subscript on ζ indicates the spatial coordinate)

$$T_\pm = \frac{1}{D_\pm}\begin{pmatrix}(1-r^2)(1+e^{2i\zeta}) & \mp 4re^{2i\zeta_1}\\ \\ \pm 4re^{2i\zeta_2} & (1-r^2)(1+e^{2i\zeta})\end{pmatrix}, \tag{3.27}$$

where

$$D_\pm = e^{2i\zeta}(1\mp r)^2 + (1\pm r)^2. \tag{3.28}$$

Note that

$$D \equiv D_\pm^* D_\pm = 2[1 + 6r^2 + r^4 + (1-r^2)^2\cos 2\zeta]. \tag{3.29}$$

A straightforward calculation now reveals (the superscript now indicates the time coordinate) that

$$\langle J_{12}^0 \rangle = -\langle J_{22}^0 \rangle = \frac{4er(\sin 2\zeta_1^{(2)} - \sin 2\zeta_2^{(2)})}{\Delta D^{(2)}} \tag{3.30}$$

which are expressed in terms of potentials at the intermediate time 2, while

$$\langle J^0_{11}\rangle = -\langle J^0_{21}\rangle = -\langle J^0_{12}\rangle^{\zeta^{(1)}}, \tag{3.31}$$

where the last notation means that the expression is the same as (3.30) except that the potential is at time 1. In the same way, we find

$$\langle J^1_{12}\rangle = \langle J^1_{22}\rangle = \frac{4er^2(\sin 2\zeta_1^{(2)} + \sin 2\zeta_2^{(2)})}{\Delta D^{(2)}} \tag{3.32}$$

and

$$\langle J^1_{11}\rangle = \langle J^1_{21}\rangle = \langle J^1_{12}\rangle^{\zeta^{(1)}}. \tag{3.33}$$

It follows immediately that the vector current is conserved:

$$\langle \text{“}\partial_\mu J^\mu\text{”}\rangle = 0. \tag{3.34}$$

However, the axial-vector current is anomalous

$$\langle \text{“}\partial_\mu J^\mu_5\text{”}\rangle = \frac{8eh}{\Delta^3}\left(\frac{\sin 2\zeta_2^{(2)}}{D^{(2)}} - \frac{\sin 2\zeta_2^{(1)}}{D^{(1)}}\right). \tag{3.35}$$

Note that this reduces to the previous $r=1$ result, (3.24).

In the same way the calculation at $M=3$ can be carried out. Here the transfer matrix is given by

$$T_\pm = \frac{1}{D_\pm}\times \tag{3.36}$$
$$\begin{pmatrix} (1-r^2)(1\pm r+(1\mp r)e^{2i\zeta}) & 4r(r\mp 1)e^{2i(\zeta_1+\zeta_3)} & 4r(r\pm 1)e^{2i\zeta_1} \\ 4r(r\pm 1)e^{2i\zeta_2} & (1-r^2)(1\pm r+(1\mp r)e^{2i\zeta}) & 4r(r\mp 1)e^{2i(\zeta_1+\zeta_2)} \\ 4r(r\mp 1)e^{2i(\zeta_2+\zeta_3)} & 4r(r\pm 1)e^{2i\zeta_3} & (1-r^2)(1\pm r+(1\mp r)e^{2i\zeta}) \end{pmatrix},$$

where

$$D_\pm = (1\pm r)^3 + (1\mp r)^3 e^{2i\zeta}. \tag{3.37}$$

Here the common denominator is

$$D = D_\pm^* D_\pm = 2[1 + 15r^2 + 15r^4 + r^6 + (1 - r^2)^3 \cos 2\zeta]. \tag{3.38}$$

We find

$$\begin{aligned}\langle J^0_{12}\rangle = \frac{4er}{\sqrt{3}D^{(2)}\Delta}\Big[&(1+3r^2)\left(\sin 2\zeta_1^{(2)} - \sin 2\zeta_2^{(2)}\right) + (1-r^2)\Big(\sin 2(\zeta_1^{(2)} + \zeta_3^{(2)})\\ &- \sin 2(\zeta_2^{(2)} + \zeta_3^{(2)})\Big)\Big]\end{aligned} \tag{3.39}$$

where the corresponding expressions for $\langle J^0_{22}\rangle$ and $\langle J^0_{32}\rangle$ are obtained by translation. Again, $\langle J^0_{11}\rangle$ is obtained by replacing $\zeta^{(2)}$ by $\zeta^{(1)}$,

$$\langle J^0_{11}\rangle = -\langle J^0_{12}\rangle^{\zeta^{(1)}}. \tag{3.40}$$

The vacuum expectation value of J^1 is

$$\begin{aligned}\langle J^1_{11}\rangle = \frac{4er^2}{\sqrt{3}D^{(1)}\Delta}\Big[&(3+r^2)\left(\sin 2\zeta_1^{(1)} + \sin 2\zeta_2^{(1)}\right) + (1-r^2)\Big(\sin 2(\zeta_1^{(1)} + \zeta_3^{(1)})\\ &+ \sin 2(\zeta_2^{(1)} + \zeta_3^{(1)}) - 2\sin 2\zeta_3^{(1)}\Big)\Big],\end{aligned} \tag{3.41}$$

and $\langle J^1_{12}\rangle$ is obtained by replacing $\zeta^{(1)}$ by $\zeta^{(2)}$,

$$\langle J^1_{12}\rangle = \langle J^1_{11}\rangle^{\zeta^{(2)}}. \tag{3.42}$$

Now, in general, neither vector nor axial-vector current is conserved:

$$\begin{aligned}\langle \text{“}\partial_\mu J^\mu\text{”}\rangle &= \frac{1}{2h}\langle J^0_{22} + J^0_{12} - J^0_{21} - J^0_{11}\rangle + \frac{1}{2\Delta}\langle J^1_{22} + J^1_{21} - J^1_{12} - J^1_{11}\rangle\\ &= \frac{2e(1-r^2)}{\sqrt{3}D^{(2)}\Delta^2}\Big[(1-r^2)\left(\sin 2\zeta_1^{(2)} - \sin 2\zeta_3^{(2)}\right) + (1+r^2)\Big(\sin 2(\zeta_1^{(2)} + \zeta_2^{(2)})\\ &\qquad - \sin 2(\zeta_2^{(2)} + \zeta_3^{(2)})\Big)\Big] + (\zeta^{(2)} \to \zeta^{(1)}),\end{aligned} \tag{3.43}$$

$$\begin{aligned}\langle \text{“}\partial_\mu J_5^\mu\text{”}\rangle &= \frac{1}{2h}\langle J^1_{22} + J^1_{12} - J^1_{21} - J^1_{11}\rangle + \frac{1}{2\Delta}\langle J^0_{22} + J^0_{21} - J^0_{12} - J^0_{11}\rangle\\ &= \frac{4er}{\sqrt{3}D^{(2)}\Delta^2}\Big[4(1+r^2)\sin 2\zeta_2^{(2)} + (1-r^2)\left(\sin 2(\zeta_1^{(2)} + \zeta_2^{(2)}) + \sin 2(\zeta_2^{(2)} + \zeta_3^{(2)}\right)\Big]\\ &\qquad -(\zeta^{(2)} \to \zeta^{(1)}).\end{aligned} \tag{3.44}$$

However, if we require the vanishing of the vector anomaly, that is, electric current conservation, we must set $r = 1$ and then the result in (3.24) follows.

IV. CONCLUSION

In this letter we have examined the axial-vector anomaly in two-dimensional electrodynamics in the context of the finite-element lattice. The results seem quite definitive: In agreement with arguments based on the dispersion relation (or the fermion propagator), chiral symmetry is broken and no fermion doubling occurs. The reader will ask, how can this be? The answer lies in the fact that in our Minkowski-space formulation, no Lagrangian exists from which the equation of motion (Dirac equation) is derivable. Hence symmetry arguments do not imply a corresponding conserved current. This conclusion is supported by the results discussed in the Appendix to [9]. There, it is shown that in Euclidean space (with periodic boundary conditions in all directions) it would be possible to define a Lagrangian, from which conserved vector and axial-vector currents could be derived. But, in Minkowski space this cannot be done, and the gauge-invariant local currents are anomalous.

ACKNOWLEDGEMENT

This paper is dedicated to the memory of Julian Schwinger, who contributed so much to the understanding of field theory, and was the discoverer of anomalies. He was not only a preëminent physicist, but a great human being, and the world has suffered a great loss in his passing. The work reported here was supported in part by the U.S. Department of Energy. I gratefully acknowledge useful conversations with Carl Bender, Dean Miller, Stephan Siegemund-Broka, and Tai Wu.

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FIGURES

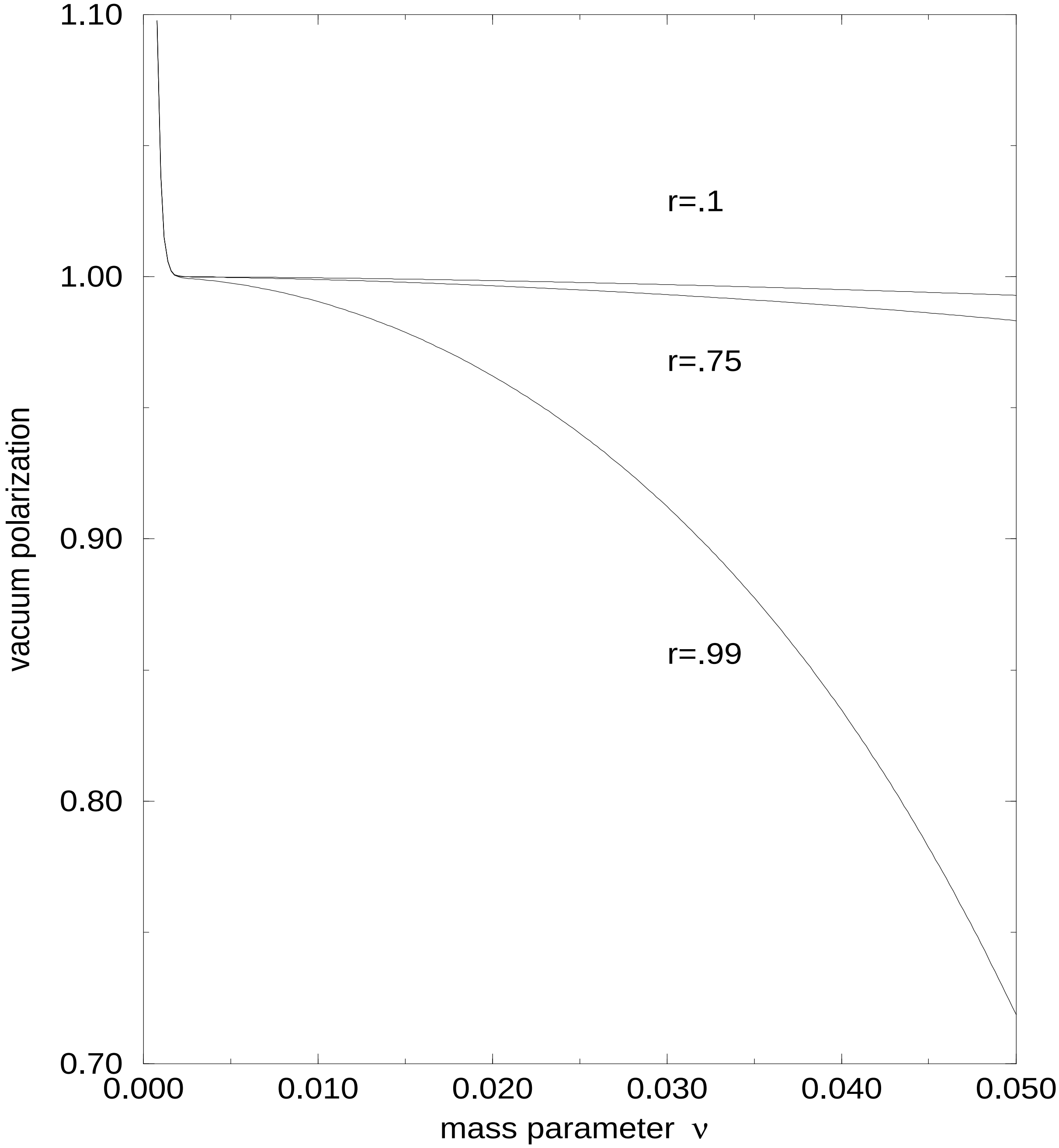

FIG. 1. Plot of the one-loop vacuum polarization on a rectangular lattice with $M = 2533$ spatial lattice sites as a function of $\nu = \mu\Delta/2$. Shown are curves with $r = h/\Delta = 0.1$, 0.75, 0.99. The quantity plotted is $-\Pi(0)/(e^2/\pi)$.